Physics 115/242
Romberg Integration

Peter Young
(Dated: May 5, 2003)

We have already discussed the trapezium rule as an approximation to the integral

$$I = \int_a^b f(x)\, dx. \tag{1}$$

For reasons that will become clear later we will write the trapezium rule for n intervals as $I_n^{(0)}$, so

$$I_n^{(0)} = h\left[\tfrac{1}{2} f_0 + f_1 + f_2 + \cdots + f_{n-1} + \tfrac{1}{2} f_n\right], \tag{2}$$

where

$$h = \frac{b-a}{n} \tag{3}$$

is the width of one interval, $f_i \equiv f(x_i)$ with $x_i = x_0 + i\,h$, and $x_0 = a, x_n = b$. We have seen in class that, for small h, the error is proportional to h^2. It turns out that if one writes an expression for the error as a power series in h then only even powers of h appear (assuming that the integrand doesn't have any singularities). (I haven't been able to find a simple way to derive this.) In other words

$$I = I_n^{(0)} + Ah^2 + Bh^4 + Ch^6 + \cdots. \tag{4}$$

If n is even we can write the analogous expression for $n/2$ intervals as

$$I = I_{n/2}^{(0)} + A(2h)^2 + B(2h)^4 + C(2h)^6 + \cdots, \tag{5}$$

where we have noted that the width of an interval is now $2h$. If we subtract 4 times Eq. (4) from Eq. (5) we eliminate the leading error, i.e. the term of order h^2. Dividing by 3 we get

$$I = \frac{4I_n^{(0)} - I_{n/2}^{(0)}}{3} - 4Bh^4 - 20Ch^6 + \cdots. \tag{6}$$

We define

$$I_n^{(1)} = \frac{4I_n^{(0)} - I_{n/2}^{(0)}}{3} \tag{7}$$

so

$$I = I_n^{(1)} + B'h^4 + C'h^6 + \cdots, \tag{8}$$

where B' and C' It is easy to show from Eqs. (2) and (7) that $I_n^{(1)}$ is just Simpson's rule for n intervals.

Clearly we can then repeat the trick (if n is a multiple of 4) by forming a linear combination of $I_n^{(1)}$ and $I_{n/2}^{(1)}$ to get rid of the h^4 term, i.e.

$$I_n^{(2)} = \frac{16I_n^{(1)} - I_{n/2}^{(1)}}{15} \tag{9}$$

which has an error of order h^6. In general, if n is a multiple of 2^k we can form iteratively a sequence of higher order approximations, $I_n^{(k)}$, where

$$I_n^{(k)} = \frac{4^k I_n^{(k-1)} - I_{n/2}^{(k-1)}}{4^k - 1}, \tag{10}$$

for $k = 1, 2, 3, \cdots$, which have an error of order h^{2k+2}.

A convenient way of carrying this out, known as Romberg integration, is as follows. Start with 1 interval and compute $I_1^{(0)}$. Then do two intervals and compute $I_2^{(0)}$ from which $I_2^{(1)}$ can be determined using Eq. (7). Then double the number of intervals again to 4 and calculate $I_4^{(0)}$, from which $I_4^{(1)}$ can be obtained using Eq. (7), and hence $I_4^{(2)}$ obtained from Eq. (9). The sequence of quantities thus obtained can be conveniently represented as a table,

n k → ↓	0	1	2	3	
1	$I_1^{(0)}$				
2	$I_2^{(0)}$	$I_2^{(1)}$			
4	$I_4^{(0)}$	$I_4^{(1)}$	$I_4^{(2)}$		
8	$I_8^{(0)}$	$I_8^{(1)}$	$I_8^{(2)}$	$I_8^{(3)}$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

in which we recall that n is the number of intervals used in the trapezium rule. The column labeled 0 contains results for the trapezium rule, and are obtained from evaluating the function at appropriate points. Each subsequent column is obtained by manipulating the data in the column to the left of it, and contains results for a higher order approximation. Note that the function values obtained for the trapezium rule for a given number of intervals n also occur in the trapezium rule for $2n$ intervals and so do not need to be recomputed.

For a given number of number of intervals $n = 2^k$ the most accurate result is the diagonal entry $I_{2^k}^{(k)}$, and so we define successive Romberg estimates by

$$R_k = I_{2^k}^{(k)}. \tag{11}$$

If the desired accuracy is ϵ then, as an empirical rule, we keep doubling the number of intervals until

$$|R_k - R_{k-1}| < \epsilon \tag{12}$$

and take R_k to be the estimate for the integral I. Frequently the error, $|I - R_k|$, turns out to be much smaller than ϵ.

As an example consider

$$I = \int_1^2 \frac{1}{x^2}\, dx = 1/2. \tag{13}$$

Using Romberg integration we obtain the following table of values

n, k =	0	1	2	3	4	5
1	0.62500000000					
2	0.53472222222	0.50462962963				
4	0.50899376417	0.50041761149	0.50013681028			
8	0.50227085033	0.50002987904	0.50000403021	0.50000192259		
16	0.50056917013	0.50000194339	0.50000008102	0.50000001833	0.50000001086	
32	0.50014238459	0.50000012275	0.50000000137	0.50000000010	0.50000000003	0.50000000002

As stated above, for a given value of n (i.e. for a given number of function evaluations), the most accurate value is for the largest value of k, i.e. the diagonal entry. These are the successive Romberg estimates R_k, i.e.

k	R_k
0	0.62500000000
1	0.50462962963
2	0.50013681028
3	0.50000192259
4	0.50000001086
5	0.50000000002

If we had wanted an accuracy of $\epsilon = 10^{-5}$ we would have stopped at $k = 4$ since $|0.50000001086 - 0.50000192259| = 0.00000191173 < 10^{-5}$ whereas Eq. (12) is not satisfied for smaller values of k. The resulting value of 0.50000001086 is indeed 1/2 to 5 decimal places. In fact it differs from the exact result by about 1.1×10^{-8}, quite a lot smaller than the desired accuracy, but it is better to be on the safe side.

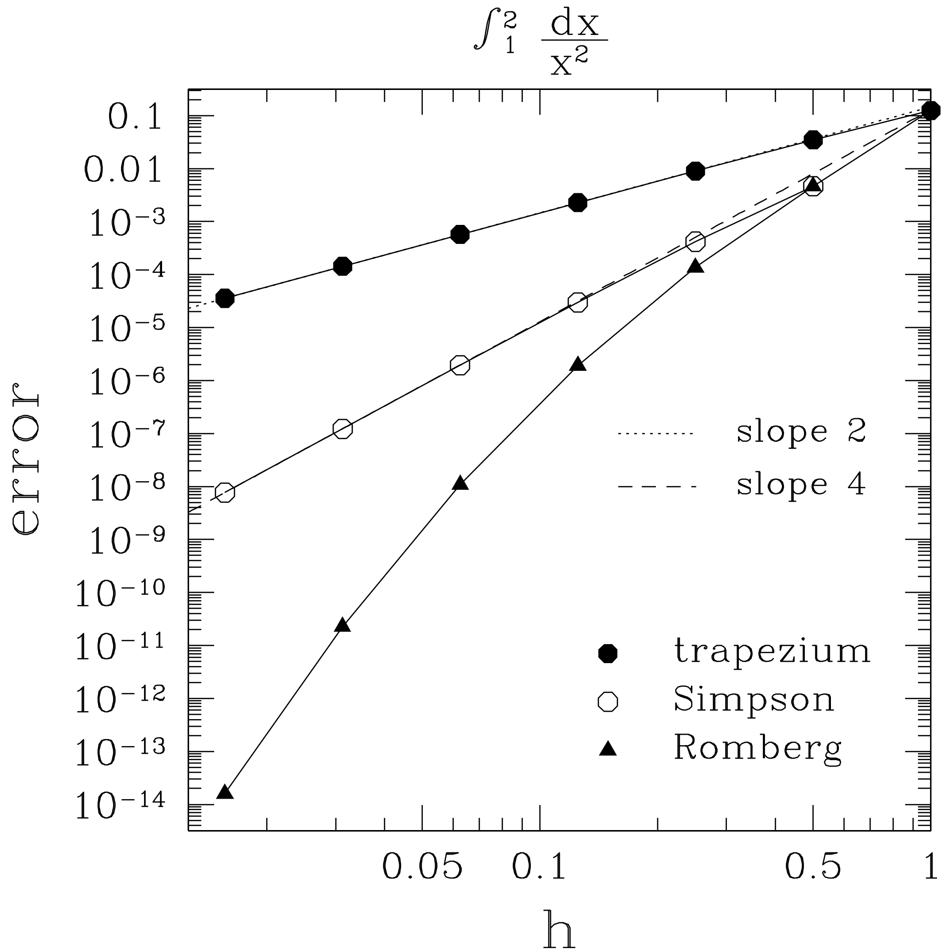
The figure shows the errors for the trapezium rule, Simpson's rule, and Romberg integration graphically (one more iteration has been done than in the table). The dashed lines indicate the leading error for the trapezium rule

$$-\frac{h^2}{12}(f'(b) - f'(a)) = \frac{7}{48}h^2, \tag{14}$$

and Simpson's rule

$$-\frac{h^4}{180}(f'''(b) - f'''(a)) = \frac{31}{240}h^4. \tag{15}$$

From the figure and these expressions for the error, we deduce that to reach a precision of 10^{-14}, close to machine accuracy with double precision, Romberg integration needs 64 intervals, while Simpson's rule needs about 1900 intervals, and the trapezium rule needs no less than 3.8×10^6 intervals.



All in all, Romberg integration is a powerful but quite simple method, which I recommend for general use. For a given number of intervals, it is much more accurate than the trapezium rule, and quite a bit more accurate than Simpson's rule, but does not need any more function evaluations.